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ZERO-MOMENT ELLIPSOIDAL DOMES

JØRGEN JUNCHER JENSEN

Department of Ocean Engineering,
The Technical University of Denmark, Lyngby, Denmark

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Abstract—This paper is concerned with the problem of designing ellipsoidal domes so that they are in a membrane state of stress. The investigation is based on a method proposed by Vekua, in which an adjustment of the gravity of the shell is used to ensure a membrane state. The theory is specialized to ellipsoidal shells and some numerical results of the gravity distribution (i.e. thickness of the shell), and the corresponding stresses and linear elastic displacements are presented.

However, it should be noted that the solutions obtained in this way are not unique.

NOMENCLATURE

$a_{\alpha\beta}$, a	metric tensor of the middle surface and its determinant
$d_{\alpha\beta}$	curvature tensor of the middle surface
f^i	Cartesian coordinates of the middle surface
$h = h(x^1, x^2)$	the thickness of the shell
p, F^α	external loadings in surface coordinates
r, θ	polar coordinates, defined through equation (2.9)
s	arclength along the boundary
u_0, u^α	bending displacement field in surface coordinates
w, w^*, w^{**}	complex functions, defined by equations (1.13, 1.18 and 1.21), respectively
(x^1, x^2)	conjugate isometric surface coordinate system
z	complex parameter, defined by equation (1.13)
A^i_j, A^i, C_i, C	transformation constants, which relate any ellipsoidal shell to a sphere
F^*, F^{**}	stress and displacement functions, defined by equations (1.18) and (1.20), respectively
$\bar{F}^i$	prescribed external loadings in Cartesian coordinates
$\bar{F}^i_t$	total external loadings in Cartesian coordinates
K	Gaussian curvature of the middle surface
$N^{\alpha\beta}$	the symmetric membrane stress tensor
N_1, N_2	the principal stresses
R	constant, defined through equation (2.8)
R_s	radius of the reference sphere
R_n, S_n, T_n, W_n^1	Fourier functions, defined through equations (2.14, 2.16 and 2.18), respectively
S, T	complex functions, defined through equation (2.13)
$\bar{T}^i$	the boundary forces in Cartesian coordinates
$\bar{U}^i$	bending displacement field in Cartesian coordinates
V	transformation function, defined by equation (2.1)
$\bar{V}^i$	displacement field, derived from Hooke's law
X^i	normal to the middle surface
$\tilde{X}^i$	vector function, defined by equation (2.6)
$\psi^i_{(J)}$	a system of orthonormal displacement fields, in Cartesian coordinates
γ	specific weight of the material of the shell
subscript c	denotes complex conjugation
subscript s	denotes quantities related to the reference sphere
∂_z, $(\ )_{,z}$	denotes partial differentiation with respect to z.

INTRODUCTION

In this paper a method developed by Vekua[1] is used to realize the membrane state of stress in domes of ellipsoidal shape. The general idea is that with an appropriate choice of the thickness variation across the shell, the gravity forces, together with the other prescribed loadings, should ensure the membrane state.

In many shell constructions the membrane state of stress is considered to be an optimal solution in the sense that the stresses are uniform through the thickness of the shell.

Two main approaches have been used previously to attain a membrane state in shells. The first method is based on the solution of the equilibrium equations together with the compatibility equations and a constitutive law. As this set of equations is hard to solve in general, all works so far have been limited to axisymmetric shells. In a recent paper by Nemirovskii and Starostin[2], the membrane state of stress for this class of shells is reached by varying one or more of the parameters: meridian shape of the middle surface, external loadings, thickness function and distribution of reinforcement. In the absence of surface forces a very clear formulation of this method is given in an early paper by Horne[3].

The second method makes use of the fact that a membrane shell is a statically determinate structure, provided the boundary conditions are suitably defined. Therefore, the equilibrium equations can be solved for the stresses without any assumptions on the stress–strain relations. The solution of the equilibrium equations can be formulated in several ways, all of which take into account the gravity of the shell. The usual procedure is to take the middle surface, thickness and "half" of the boundary tractions as prescribed and then solve for the stresses, which ultimately gives the rest of the tractions. Another way is to prescribe a stress distribution and then solve for the thickness and the middle surface (e.g. domes of uniform strength). Both these formulations have been thoroughly treated in earlier papers [4, 7]. In the method proposed by Vekua[1], tractions are prescribed completely on the boundary. Furthermore, if the middle surface and the surface forces, except the gravity forces, are taken as given, the equilibrium equations can be solved for the stresses *and* the thickness distribution. The thickness is determined in such a way that *a* stress distribution can be found which satisfies the prescribed tractions. It should be emphasized, however, that with this method, the solution is not unique. The last-mentioned method, which does not seem to have been used previously, turns out to be a very useful method in the design of ellipsoidal domes loaded by a *single* load system, such as e.g. snow, hydrostatic pressure and/or concentrated forces. However, as the geometry of the dome normally would vary with the type or direction of the applied loads, it would in general not be possible to find a bending-free design in the case of a *multiple* load system, such as, for example, wind blowing from different directions.

MEMBRANE THEORY

This section is devoted to a short outline of parts of the membrane theory of shells derived by Vekua[1].

The Vekua theorem

In the membrane state of stress the equilibrium equations for a thin shell read

$$\begin{aligned} D_\alpha N^{\alpha\beta} + F^\beta &= 0 \\ d_{\alpha\beta} N^{\alpha\beta} + p &= 0 \end{aligned} \tag{1.1}$$

where $N^{\alpha\beta}$ is the symmetric stress tensor and $d_{\alpha\beta}$ is the symmetric tensor of curvature. F^{β} and p are related to the cartesian components $\bar{F}_t^i$ of the surface loading through

$$\bar{F}_t^i = f^i_{,\alpha}F^{\alpha} + pX^i \tag{1.2}$$

where comma means partial differentiation. Furthermore, D_{α} denotes covariant differentiation, $f^i_{,\alpha}$ are the base vectors of the surface coordinate system and X^i is the normal to the surface. Here and in the following, the summation convention is adopted.

Now, let $\bar{U}^i$ denote an arbitrary bending displacement field

$$\begin{gathered} \bar{U}^i = f^i_{,\alpha}u^{\alpha} + X^i u_0 \\ \tfrac{1}{2}(D_{\alpha}u_{\beta} + D_{\beta}u_{\alpha}) - d_{\alpha\beta}u_0 = 0 \end{gathered} \tag{1.3}$$

and let $\bar{T}^i$ be the boundary forces, satisfying

$$\bar{T}^i X^i = 0 \tag{1.4}$$

along the boundary L.

The Vekua theorem[1] then states that

$$\iint_S \bar{F}_t^i \bar{U}^i \,\mathrm{d}S + \int_L \bar{T}^i \bar{U}^i \,\mathrm{d}s = 0 \tag{1.5}$$

is a necessary and sufficient condition for the existence of a membrane state of stress in a shell with positive Gaussian curvature throughout. Gol'Denveizer and Zveriaev[5] have shown that this theorem can be extended to shells with zero Gaussian curvature, provided that the boundaries are not along the lines with infinite radius of curvature.

Determination of thickness distribution

From the complete solution $\bar{U}^i_{(j)}$ of equation (1.3) we construct a set of orthonormal functions $\psi^i_{(j)}$ in the sense that (δ_{jk} being Kronecker's symbol)

$$\iint_S \psi^3_{(j)}\psi^3_{(k)} \,\mathrm{d}S = \delta_{jk}. \tag{1.6}$$

This can be achieved by the Gram–Schmidt orthogonalization procedure, which yields

$$\psi^i_{(j)} = c_j\left\{\bar{U}^i_{(j)} - \sum_{n=1}^{j-1} I_{jn}\psi^i_{(n)}\right\}, \qquad j = 1, 2, 3, \ldots \tag{1.7}$$

where

$$I_{jn} \equiv \iint_S U^3_{(j)}\psi^3_{(n)} \,\mathrm{d}S$$

$$c_j = \left\{\iint_S \left\{\bar{U}^3_{(j)} - \sum_{n=1}^{j-1} I_{jn}\psi^3_{(n)}\right\}^2 \mathrm{d}S\right\}^{-1/2}.$$

The surface loadings are separated into

$$\bar{F}_t^i = \bar{F}^i - \gamma h \delta^i_3 \tag{1.8}$$

where $\bar{F}^i$ are the prescribed surface forces, h the thickness of the shell, and γ the weight per unit volume. Using equations (1.6) and (1.8), equation (1.5) yields

$$\gamma h = \sum_{j=1}^{\infty} k_j \psi^3_{(j)} + \psi \tag{1.9}$$

where

$$k_j = \iint_S \bar{F}^i \psi^i_{(j)} \, \mathrm{d}S + \int_L \bar{T}^i \psi^i_{(j)} \, \mathrm{d}s$$

and where ψ is *any* function that satisfies

$$\iint_S \psi \psi^3_{(j)} \, \mathrm{d}S = 0 \qquad j = 1, 2, 3, \ldots \tag{1.10}$$

As the introduction of gravity forces does not alter the horizontal force balance and the moment equilibrium about a vertical axis, the necessary conditions for achieving a membrane state by an adjustment of the gravity forces are

$$\begin{aligned} &\iint_S \bar{F}^\alpha \, \mathrm{d}S + \int_L \bar{T}^\alpha \, \mathrm{d}s = 0 \qquad \alpha = 1, 2 \\ &\iint_S \bar{F}^\alpha e_{\alpha\beta} f^\beta \, \mathrm{d}S + \int_L \bar{T}^\alpha e_{\alpha\beta} f^\beta \, \mathrm{d}s = 0 \qquad \alpha, \beta = 1, 2 \end{aligned} \tag{1.11}$$

together with equation (1.4) and a solution γh that is positive everywhere. $e_{\alpha\beta}$ is the alternating symbol. Vekua[1] has also shown that these rigid body equilibrium equations are sufficient conditions.

As shown by Vekua[1] the equations (1.3) can be rewritten in a more convenient form by introducing a conjugate isometric surface coordinate system (x^1, x^2) in which†

$$d_{\alpha\beta} = \sqrt{Ka} \; \delta_{\alpha\beta} \tag{1.12}$$

where K is the Gaussian curvature and a is the determinant of the metric tensor. By introducing the complex functions

$$\begin{aligned} z &= x^1 + ix^2 \\ w &= a^{-1/2} K^{-1/4}(u_1 + iu_2) \end{aligned} \tag{1.13}$$

where i is the imaginary unit, it can be shown that

$$\bar{U}^i_{(j)} = Re\{-2K^{-1/4} w_j X^i_{,z} + K^{-1} a^{-1/2} (a^{1/2} K^{3/4} w_j)_{,z} X^i\} \tag{1.14}$$

where w_j is the jth solution of‡

$$\partial_{z_c} w + B w_c = 0. \tag{1.15}$$

The derivatives are defined by

$$\partial_z \equiv (\;)_{,z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2}\right) \tag{1.16}$$

† X^i is then directed to the convex side.

‡ A subscript c means complex conjugation of a function.

and furthermore

$$B=\frac{1}{4}\left(\begin{Bmatrix} 1 \\ 2 \quad 2\end{Bmatrix}-\begin{Bmatrix} 1 \\ 1 \quad 1\end{Bmatrix}+2\begin{Bmatrix} 2 \\ 1 \quad 2\end{Bmatrix}\right)-\frac{i}{4}\left(\begin{Bmatrix} 2 \\ 1 \quad 1\end{Bmatrix}-\begin{Bmatrix} 2 \\ 2 \quad 2\end{Bmatrix}+2\begin{Bmatrix} 1 \\ 1 \quad 2\end{Bmatrix}\right)$$

$\begin{Bmatrix} \alpha \\ \beta \quad \gamma\end{Bmatrix}$ being the Christoffel symbols. In deriving equation (1.14) we have made use of Godazzi's equations and the well known identity

$$\frac{\partial \log(\sqrt{a})}{\partial x^{\alpha}}=\begin{Bmatrix} \beta \\ \alpha \quad \beta\end{Bmatrix}.$$

The determination of $\overline{U}^{i}_{(j)}$ is now reduced to the solution of equation (1.15). This equation belongs to the class of generalized Cauchy–Riemann equations, which have been thoroughly treated by Vekua[1].

With a transformation like that used to solve equation (1.3), equations (1.1) can be written as

$$\begin{aligned} N^{11}-iN^{12}&=\frac{w^*}{aK^{1/4}}-\frac{p}{2\sqrt{aK}} \\ N^{11}+N^{22}&=-\frac{p}{\sqrt{aK}} \end{aligned} \tag{1.17}$$

where w^* is the unique solution of

$$\begin{aligned} \partial_{z_c}w^*-B_c w_c^*&=F^* \\ F^*&=\tfrac{1}{2}a^{1/2}K^{1/4}\left\{K^{1/2}\left(\frac{p}{K}\right)_{,z}-\sqrt{a}(F^1-iF^2)\right\} \end{aligned} \tag{1.18}$$

with the boundary condition

$$w^*=-\frac{2i}{z'K^{1/4}}\overline{T}^i X^i_{,z}-\frac{1}{2}\frac{p\sqrt{a}}{K^{1/4}}\frac{z_c'}{z'} \quad \text{on} \quad L \tag{1.19}$$

where prime denotes differentiation with respect to the arc length on the boundary. By a similar transformation, the linear stress–strain relations can be written as

$$\begin{aligned} \partial_{z_c}w^{**}+Bw_c^{**}&=F^{**} \\ F^{**}&=\frac{1+\nu}{2Eh}a^{-1/2}K^{-1/4}\left\{N_{11}-N_{22}+2iN_{12}-\frac{\nu}{1+\nu}(a_{11}-a_{22}+2ia_{12})N^{\alpha}_{\alpha}\right\} \end{aligned} \tag{1.20}$$

where E is Young's modulus and ν Poisson's ratio. The displacements $\overline{V}^i$ are then given by

$$\begin{aligned} w^{**}&=a^{-1/2}K^{-1/4}(v_1+iv_2) \\ v_0&=K^{-1}a^{-1/2}Re\{(a^{1/2}K^{3/4}w^{**})_{,z}\}-\tfrac{1}{2}a^{-1/2}K^{-1/2}E_{rr} \\ E_{rr}&=\frac{1}{Eh}\{(1+\nu)N_{rr}-\nu a_{rr}N^{\alpha}_{\alpha}\} \\ \overline{V}^i&=f^i_{,\alpha}v^{\alpha}+X^i v_0 \end{aligned} \tag{1.21}$$

ELLIPSOIDAL SHELLS

The solutions of the generalized Cauchy–Riemann equations (1.15, 1.18 and 1.20) are especially simple when $B \equiv 0$. Vekua[1, 6] has shown that this is the case if and only if the middle surface of the shell is of second order (i.e. an ellipsoidal shell). Taking a sphere (f_s^i) as reference we can obtain any ellipsoidal dome (f^i) by the projective transformation†

$$f^i = \frac{A_j^i f_s^j + A^i}{V}; \; V = C_k f_s^k + C; \; D = \begin{vmatrix} A_1^1 & A_1^2 & A_1^3 & C_1 \\ A_2^1 & A_2^2 & A_2^3 & C_2 \\ A_3^1 & A_3^2 & A_3^3 & C_3 \\ A^1 & A^2 & A^3 & C \end{vmatrix} \neq 0. \tag{2.1}$$

By a suitable choice of the radius R_s of the reference sphere, the determinant D can be taken to unity without losing generality, and then (see Ref. [6])

$$K = \frac{1}{V^8}\frac{a_s^2}{a^2} K_s \tag{2.2}$$

where subscript s refers to quantities related to the sphere. Furthermore, the conjugate isometric coordinate system is invariant with respect to the transformation (2.1) and for the sphere we find in this system

$$\begin{aligned} f_s^i &= \frac{R_s}{1 + zz_c}(2x^1, 2x^2, 1 - zz_c) \\ a_s &= \frac{16R_s^4}{(1 + zz_c)^4} \\ X_s^i &= -f_s^i/R_s \\ K_s &= R_s^{-2}. \end{aligned} \tag{2.3}$$

As basic solutions w_j to equation (1.15) we choose

$$w_j = \begin{cases} z^{(j+1)/2} & j = 1, 3, 5, \ldots \\ iz^{j/2} & j = 2, 4, 6, \ldots \end{cases} \tag{2.4}$$

and using equations (2.2–2.4), equation (1.14) yields

$$\overline{U}_{(j)}^i - i\overline{U}_{(j+1)}^i = -\frac{2}{V} z^{(j+1)/2}\tilde{X}_{,z}^i + \frac{1}{V} z^{(j-1)/2}\left(\frac{j-3}{2} + \frac{2}{1 + zz_c}\right)\tilde{X}^i \quad j = 1, 3, 5, \ldots \tag{2.5}$$

where

$$\tilde{X}^i \equiv V^3 \sqrt{\frac{a}{a_s}}\, X^i. \tag{2.6}$$

From equation (2.1) we obtain

$$f_{,\alpha}^i = \frac{1}{V} p_j^i f_{s,\alpha}^j \tag{2.7}$$

† Actually we do not need V, but this function may be useful in the numerical computations.

where

$$p^i_j = \frac{1}{V}\{(A^i_j C_l - A^i_l C_j)f^l_s + (CA^i_j - A^i C_j)\}.$$

This yields

$$\tilde{X}^1 = e_{lmk} p^2_m p^3_k V X^l_s$$

and similar formulae for the two other components of $\tilde{X}^i$. e_{jmk} is the $3 \times 3 \times 3$ alternating symbol. $\bar{U}^i_{(j)}$ are thereby expressed in terms of quantities belonging to the sphere and the transformation constants.

In the following we consider ellipsoidal domes without holes, i.e. the surface S of the dome is the part of the ellipsoidal shell lying above the plane $f^3 = 0$. From equations (2.1) and (2.3) we find that S is mapped into

$$\begin{cases}(x^1 - x^1_0)^2 + (x^2 - x^2_0)^2 \le R^2 & \text{when} \quad A^3_3 - A^3/R_s > 0 \\ (x^1 - x^1_0)^2 + (x^2 - x^2_0)^2 \ge R^2 & \text{when} \quad A^3_3 - A^3/R_s < 0\end{cases} \tag{2.8}$$

assuming that V is positive everywhere in the domain. The constants are

$$x^\alpha_0 = \frac{A^3_\alpha}{A^3_a - A^3/R_s}, \quad \alpha = 1, 2$$

$$R^2 = \frac{(A^3_1)^2 + (A^3_2)^2 + (A^3_3)^2 - (A^3/R_s)^2}{(A^3_3 - A^3/R_s)^2}.$$

As the domain S is circular we introduce polar coordinates

$$\begin{cases}x^1 = x^1_0 + r\cos\theta \\ x^2 = x^2_0 + r\sin\theta\end{cases} \qquad 0 \le r \le R; \qquad 0 \le \theta < 2\pi. \tag{2.9}$$

The differential surface element

$$\mathrm{d}S = a^{1/2}\,\mathrm{d}x^1\,\mathrm{d}x^2 = a^{1/2} r\,\mathrm{d}r\,\mathrm{d}\theta \tag{2.10}$$

where the determinant of the metric tensor a can be determined from equations (2.6) and (2.3). The differential line element on the boundary

$$\mathrm{d}s = R(a_{11}\sin^2\theta + a_{22}\cos^2\theta - 2a_{12}\sin\theta\cos\theta)^{1/2}\,\mathrm{d}\theta \tag{2.11}$$

where $a_{\alpha\beta}$ is the metric tensor evaluated on the boundary. This tensor is calculated from equation (2.7). With prescribed external loadings $\bar{F}^i$, $\bar{T}^i$ and prescribed geometry (i.e. A^i_j, A^i, C_j, C and R_s), the procedure outlined in the previous chapter yields one value of the function γh which ensures a membrane state of stress in the dome. In the infinite summation in equation (1.9) we take as many terms as necessary to obtain a reasonable accuracy.

Bearing in mind the definition (1.16) for the derivatives with respect to z and the relation (2.9), we can rewrite equation (1.18) as

$$\frac{\partial w^*}{\partial r} + \frac{i}{r}\frac{\partial w^*}{\partial \theta} = 2F^*(r, \theta)\mathrm{e}^{-i\theta}. \tag{2.12}$$

In order to avoid numerical differentiations of F^*, it is expedient to separate this into

$$F^*(r, \theta) = T(r, \theta) + S_{,z}(r, \theta) \tag{2.13}$$

where, from equations (1.2, 1.18 and 2.3),

$$T(r,\theta) = R_s^{1/2}\frac{a^{1/2}}{V}\left(\bar{F}_t^i\tilde{X}^i_{,z} + \frac{z_c}{1+zz_c}\bar{F}_t^i\tilde{X}^i\right)$$

$$S(r,\theta) = \tfrac{1}{2}R_s^{1/2}\frac{a^{1/2}}{V}\bar{F}_t^i\tilde{X}^i.$$

The solution w^* is obtained by substitution of

$$w^*(r,\theta) = \sum_{n=-\infty}^{\infty} R_n(r)\mathrm{e}^{in\theta} \tag{2.14}$$

into equation (2.12). Using the orthogonality properties of $\mathrm{e}^{in\theta}$ we find

$$R_n'(r) - \frac{n}{r}R_n(r) = 2\left\{T_{n+1}(r) + \frac{1}{2}\left(S_{n+2}'(r) + \frac{n+2}{r}S_{n+2}(r)\right)\right\} \tag{2.15}$$

where prime denotes differentiation with respect to r and where

$$\begin{aligned} T_n(r) &= \int_0^{2\pi} T(r,\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta \\ S_n(r) &= \int_0^{2\pi} S(r,\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta. \end{aligned} \tag{2.16}$$

Multiplying equation (2.15) by r^{-n} and integrating, we end up with

$$R_n(r) = S_{n+2}(r) + \begin{cases} 2\displaystyle\int_0^r \left(T_{n+1}(\xi) + \frac{n+1}{\xi}S_{n+2}(\xi)\right)\left(\frac{r}{\xi}\right)^n \mathrm{d}\xi; & n<0 \\ w_n^1\left(\dfrac{r}{R}\right)^n - 2\displaystyle\int_r^R \left(T_{n+1}(\xi) + \frac{n+1}{\xi}S_{n+2}(\xi)\right)\left(\frac{r}{\xi}\right)^n \mathrm{d}\xi; & n\geq 0 \end{cases} \tag{2.17}$$

where

$$\begin{aligned} w_n^1 &= \int_0^{2\pi} w_1^*(R,\theta)\mathrm{e}^{-in\theta}\,\mathrm{d}\theta \\ w_1^*(R,\theta) &= -\frac{2i}{z'K^{1/4}}\bar{T}^iX^i_z \end{aligned} \tag{2.18}$$

as the second term on the right-hand side of the boundary condition (1.19) has been included elsewhere in equation (2.17). The solution $w^*(r,\theta)$ is thereby determined and from equation (1.17) we easily find the stress tensor $N^{\alpha\beta}$. Then the principal stresses N_1, N_2 are calculated from the formula

$$\left.\begin{matrix}N_1\\N_2\end{matrix}\right\} = \tfrac{1}{2}N^\gamma_\gamma \pm \{(\tfrac{1}{2}N^\gamma_\gamma)^2 - aN\}^{1/2} \tag{2.19}$$

where

$$N \equiv N^{11}N^{22} - (N^{12})^2.$$

As seen from equations (1.20–1.21), the associated linear elastic displacements are determined in the same way as the stresses. As boundary condition we must choose prescribed inplane displacements, whereby we get a known value of w^{**} on the boundary.

NUMERICAL RESULTS

The method has been applied to a number of ellipsoidal domes with various types of surface loads and boundary tractions.

In the case of axisymmetric domes exposed to gravity forces only, the thickness was found to be constant throughout the shell. As a test, the calculated stresses were compared with the exact expressions given by Novozhilov[4], and full agreement was found.

In Figs. 1–4, the thickness, stress and displacement distributions are shown for an ellipsoidal dome with three different semiaxes, one of which is in the vertical direction. In these examples the boundary traction

$$\bar{T}^i = -\bar{n}^i \tag{3.1}$$

$\bar{n}^i$ is the unit normal to the boundary, directed outwards. Furthermore, the surface loads are taken to be the gravity forces and a hydrostatic pressure p

$$\bar{F}^i_t = -pX^i - \gamma h \delta^i_3 \tag{3.2}$$

where p takes the values

$$p = -1, 0, 1. \tag{3.3}$$

A positive sign on p means internal pressure. In Figs. 1, 3 and 4 the results for the reference sphere are also shown.

Figure 1 shows the variation of γh along the two vertical planes of symmetry and Fig. 2 shows the ground-surface in both the undeformed and the deformed state (with $\nu = 0{\cdot}3$). It will be seen that the thickness variation changes when a hydrostatic pressure is present and also that the position of the minimum thickness depends on the direction of the pressure. In Figs. 3 and 4 the associated principal stresses

$$\left(\frac{\sigma_1}{\gamma}, \frac{\sigma_2}{\gamma}\right) = \left(\frac{N_1}{\gamma h}, \frac{N_2}{\gamma h}\right) \tag{3.4}$$

are shown.

Other ellipsoidal domes with three different semiaxes show, in principle, the same characteristic as that above. However, when none of the semiaxes is vertical, the traction $\bar{T}^i$ should be prescribed otherwise than by equation (3.1) to ensure the force equilibrium (1.11). As an example the cross-section in the plane of symmetry is shown in Fig. 5. The boundary force is taken to

$$\bar{T}^i = -\bar{n}^i + q \sin \Theta \cdot \bar{t}^i \tag{3.5}$$

where Θ is the horizontal angle from the plane of symmetry $\bar{t}^i$ is a unit tangential vector to the boundary, and where the constant q takes a value that ensures the force balance (1.11).

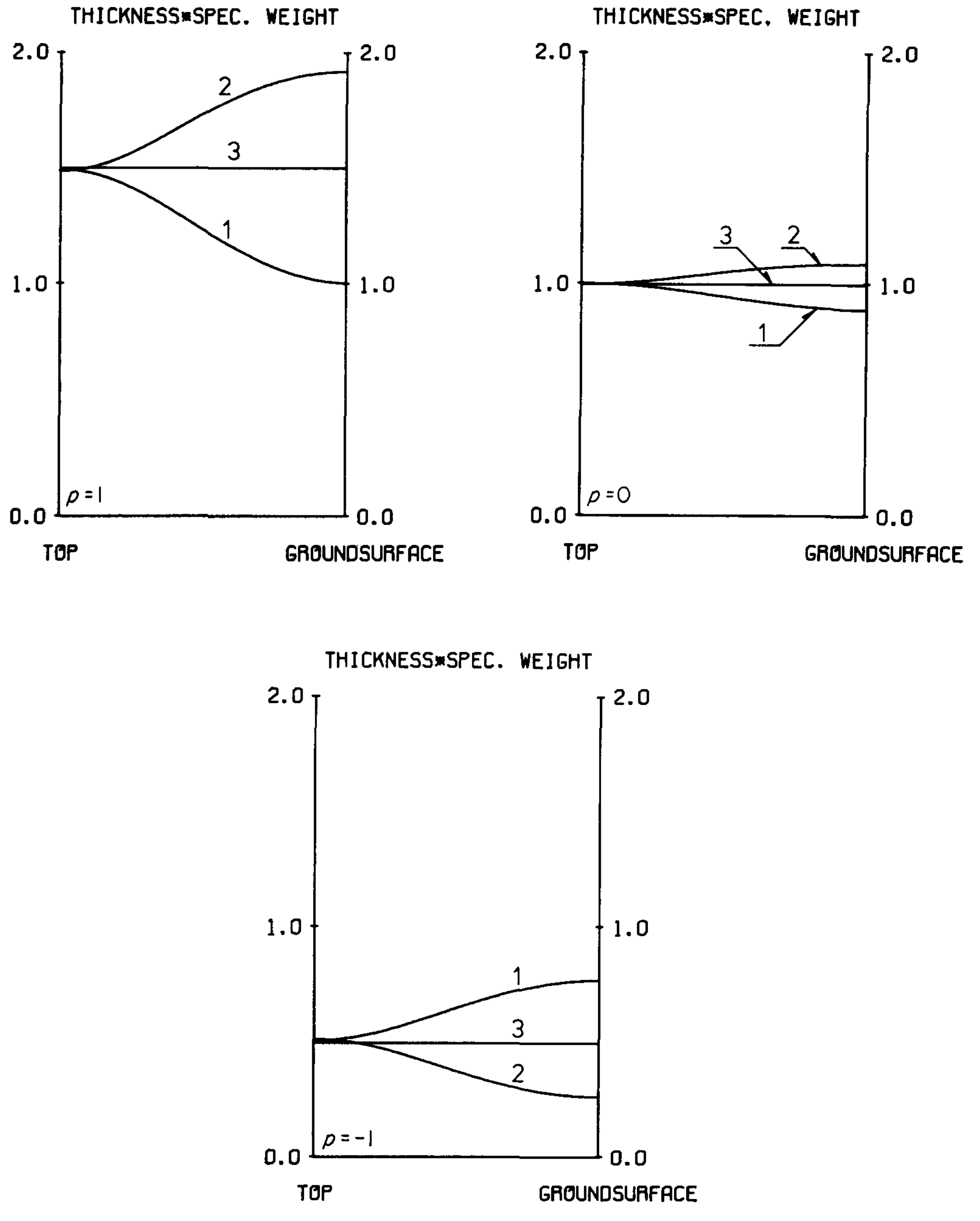


Fig. 1. Calculated thickness variation for three different values of the pressure p. Semiaxes (1·1, 0·90909, 1·0). 1 and 2: variation through the planes of symmetry. 3: variation through the reference sphere.

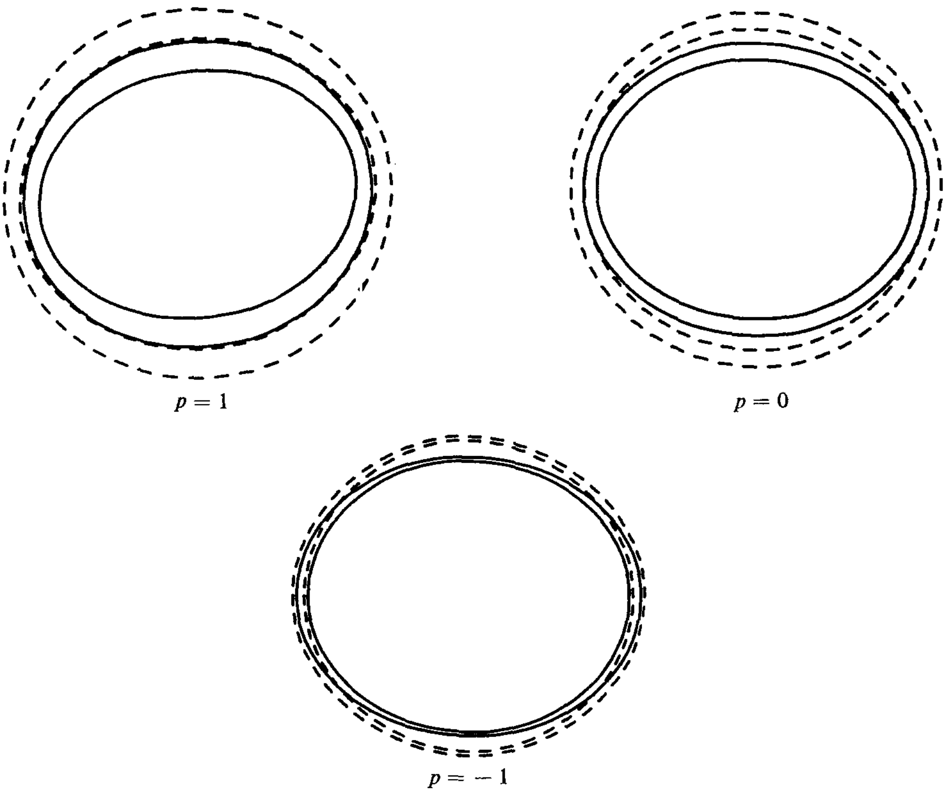
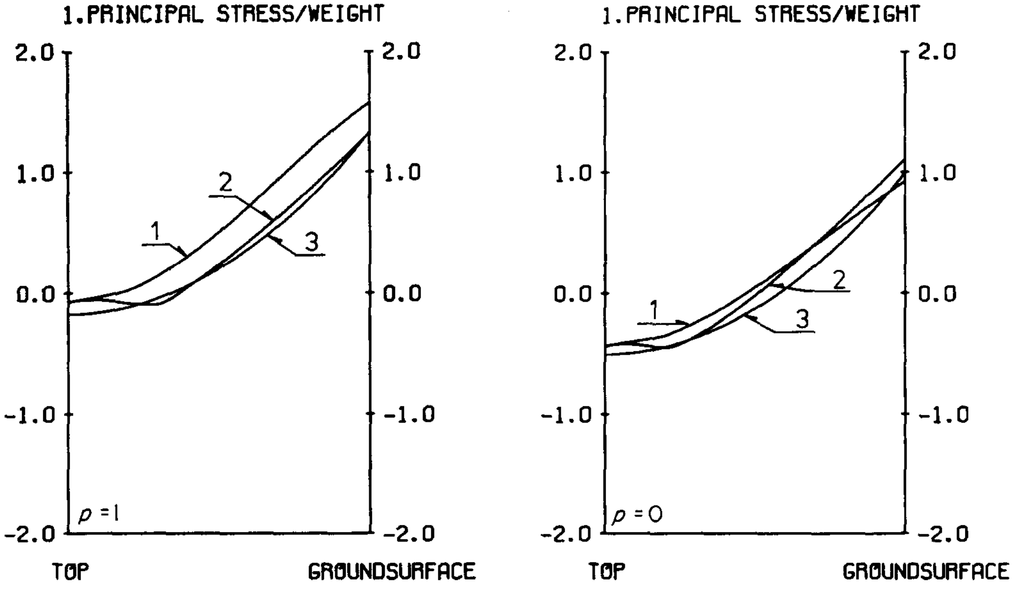


Fig. 2. Corresponding cross-sections of the shells at $f^3 = 0$. Solid lines: underformed state, dashed lines: deformed states.



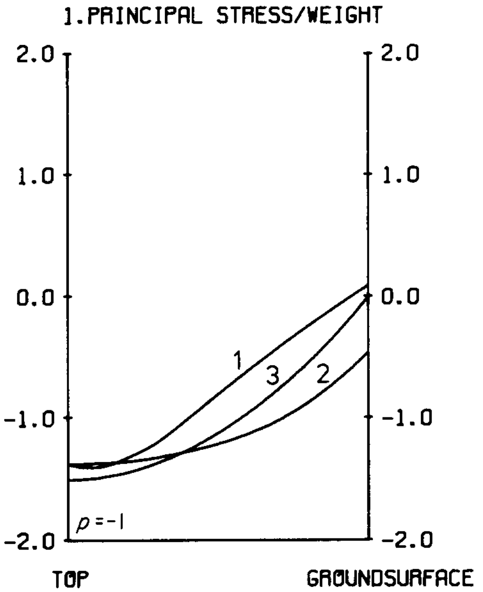
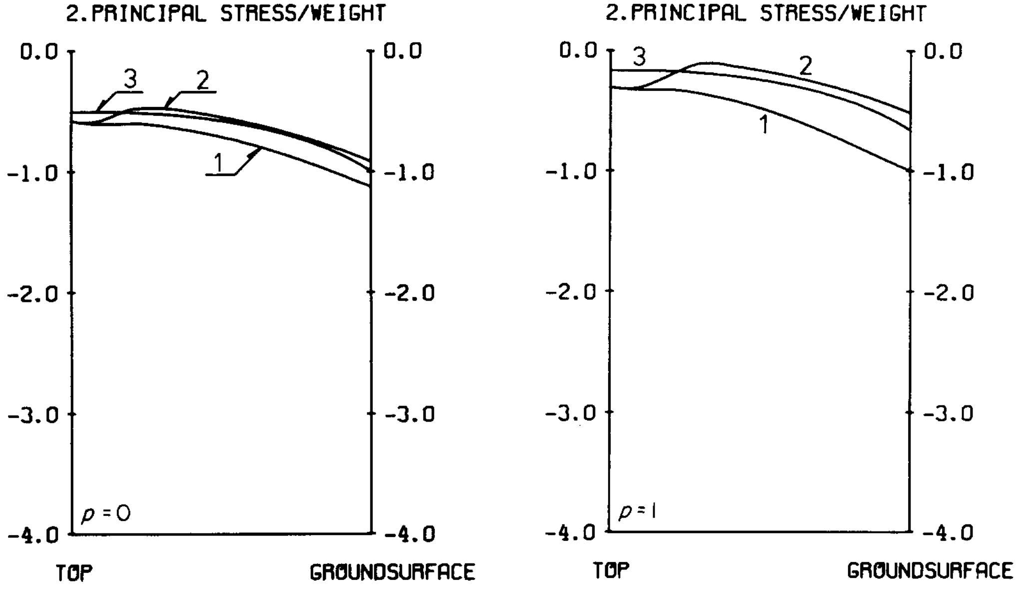


Fig. 3. Corresponding hoop stresses divided by the specific weight.



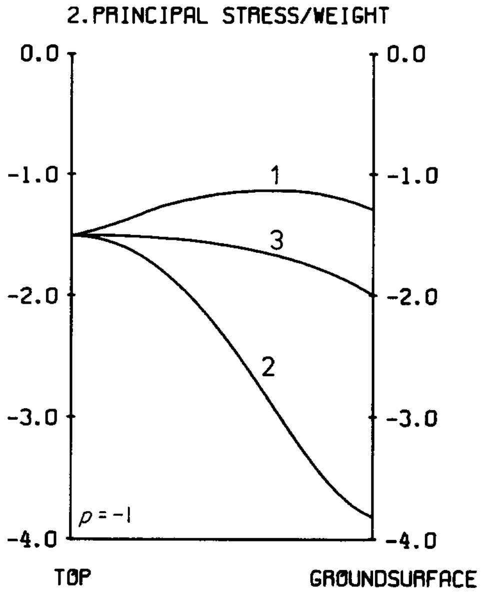


Fig. 4. Corresponding meridional stresses divided by the specific weight.

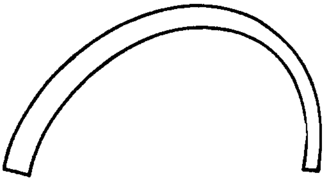


Fig. 5. An oblique dome.

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Абстракт — Работа касается задачи расчета эллипсоидальных куполов в мембранном состоянии. Исследование основано на методе Векуа, в котором используется регулирование силы тяжести, в целью гарантировки мембранного состояния. Теория ограничивается к эллипсоидальным оболочкам. Даются некоторые численные результаты для распределения силы тяжести (т. е. толщины оболочки) и соответствующие напряжения и линейные упругие перемещения.

Надо однако заметить, что полученные этим способом решения не единствены.